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Parametric (quasi-Cerenkov) x-ray free electron lasers

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Abstract. The free electron laser on the basis of parametric (quasi-Cerenkov) instability is considered. Threshold conditions for this process are obtained. It is shown that rather rigid requirements for the particle beam are needed in the x-ray region.

1. Introduction

Different mechanisms of induced x-ray radiation, which may serve as a basis for construction of an x-ray laser, functioning over the 10–100 keV range, are now being actively studied. Various types of free electron lasers, using the Compton scattering of a light wave by an electron beam [1], resonance transition radiation [2], channelled particle radiation [3, 4] have been considered. Analysis of the problem showed that to realize the induced radiation mode, rather large current densities $j > 10^{13}$ A cm^{-2} of electron relativistic beams are necessary. As a result, this points to the possibility itself of carrying out the coherent generation over the given range of x-ray radiation in question.

In previous work [5, 6] a new mechanism of spontaneous radiation over the x-ray range—parametric x-ray (quasi-Cerenkov) radiation from relativistic electrons in crystals was theoretically predicted and experimentally discovered (see [7–9]). It was shown that, although the typical refractive index of matter over the x-ray range $n = 1 - \tilde{\omega}_L^2/2\omega^2 < 1$ ($\tilde{\omega}_L$ is the Langmuir frequency, ω is the quantum frequency) the phenomenon of x-ray dynamic diffraction leads to the situation where the refractive index of x-ray quanta in a crystal may be more than unity. As a consequence, when the phase velocity of an electromagnetic wave is smaller than the light speed, this results in the Vavilov–Cerenkov effect in the x-ray region.

In addition to the spontaneous radiation, there is also induced radiation [10, 11]. It is a matter of principle that in the case of induced parametric x-ray radiation, a crystal not only leads to the existence of the phenomenon itself, but also serves as a cavity, providing the emergence of a distributed feedback. Here we deal with two substantially different cases: (i) the one-dimensional case, when the emitted and reflected waves in a lattice move along one axis, and (ii) when diffracted quanta move at a large angle with the beam velocity. In the second case the gain dependence on the beam current changes radically in comparison with case (i).

In the present work the theory of an induced parametric x-ray radiation in a crystal plate is developed. Thresholds of generation of the induced PXR are found. It is shown that the generation threshold depends strongly on the feedback geometry. The threshold, as the diffracted quanta propagate at a large angle to the beam velocity (three-dimensional cavity), may be considerably lower than for one-dimensional feedback geometry [12].

2. Theory of induced PXR in a crystal plate (general formulae)

Let an ultra-relativistic electron (positron) beam of velocity $\boldsymbol{u}$ enter, at a certain angle, a crystal plane-parallel plate with length L (the z axis is perpendicular to the crystal surface, and the plate lies in the interval $0 < z < L$). The set of equations describing the interaction of an electromagnetic wave with the 'crystal–beam' system, consists of Maxwell's equations and those of particle motion in the electromagnetic field. The dielectric susceptibility of a crystal has the form $\varepsilon(\boldsymbol{r}; \omega) = \sum_\tau \varepsilon_\tau(\omega) \exp(-\mathrm{i}\boldsymbol{\tau r})$, where τ is the reciprocal lattice vector. Perturbations of the current density and charge density in the linear field approximation may be written in the form:

$$\begin{aligned}\delta j(\boldsymbol{k}; \omega) &= e \sum_\alpha \exp(-\mathrm{i}\boldsymbol{k r}_{\alpha 0})\{\delta \boldsymbol{v}_\alpha(\omega - \boldsymbol{k u}) \\ &\quad - \mathrm{i}\boldsymbol{u}[\boldsymbol{k}\delta \boldsymbol{r}_\alpha(\omega - \boldsymbol{k u})]\}\delta \mathrm{n}(\boldsymbol{k}; \omega) \\ &= e \sum_\alpha \exp(-\mathrm{i}\boldsymbol{k}\delta \boldsymbol{r}_{\alpha 0})\{-\mathrm{i}[\boldsymbol{k r}_\alpha(\omega - \boldsymbol{k u})]\}\end{aligned} \quad (1)$$

where $\delta j(\boldsymbol{k}; \omega)$ and $\delta n(\boldsymbol{k}; \omega)$ are Fourier transformations of the expressions

$$j(\boldsymbol{r}; t) = e \sum_\alpha \boldsymbol{v}_\alpha(t)\delta[\boldsymbol{r} - \boldsymbol{r}_\alpha(t)]$$

$$n(\boldsymbol{r}; t) = \sum_\alpha \delta[\boldsymbol{r} - \boldsymbol{r}_\alpha(t)].$$

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$\boldsymbol{u}$ is the unperturbed electron (positron) velocity; $\delta \boldsymbol{v}_\alpha$ and $\delta \boldsymbol{r}_\alpha$ are perturbations of the velocity and radius vectors respectively, due to the interaction with the radiation field:

$$\boldsymbol{v}_\alpha(t) = \boldsymbol{u} + \delta \boldsymbol{v}_\alpha(t)$$

$$\boldsymbol{r}_\alpha(t) = \boldsymbol{r}_{\alpha 0} + \boldsymbol{u}t + \delta \boldsymbol{r}_\alpha(t).$$

The subscript α denotes the number of the particle.

Deriving the perturbations of velocity and radius vectors by the particle motion equation and using (1) for the current density we can obtain a set of Maxwell's equations which describes the interaction of an electromagnetic wave with a crystal, and a particle beam penetrating through it in the following form:

$$k_{\tau'}^2 \boldsymbol{E}(\boldsymbol{k}_{\tau'}, \omega) - \boldsymbol{k}_{\tau'}[\boldsymbol{k}_{\tau'} \boldsymbol{E}(\boldsymbol{k}_{\tau'}, \omega)] - \frac{\omega^2}{c^2} \sum_{\tau} \varepsilon_\tau(\boldsymbol{k}_{\tau'}, \omega) \boldsymbol{E}(\boldsymbol{k}_{\tau+\tau'}, \omega)$$
$$= -\frac{\omega_L^2}{\gamma c^2} \boldsymbol{E}(\boldsymbol{k}_{\tau'}, \omega) - \left(\frac{\omega_L^2 \boldsymbol{k}_{\tau'}}{\gamma c^2(\omega - \boldsymbol{k}_{\tau'} \boldsymbol{u})} + \frac{\omega_L^2 (k_{\tau'} C^2 - \omega^2)}{\gamma C^4 (\omega - \boldsymbol{k}_{\tau'} \boldsymbol{u})^2} \right) \times \left[(\boldsymbol{u} \boldsymbol{E}(\boldsymbol{k}_{\tau'}, \omega)) - \left(\frac{\omega_L^2 \boldsymbol{u}}{\gamma c^2 (\omega - \boldsymbol{k}_{\tau'} \boldsymbol{u})} \right) (\boldsymbol{k}_{\tau'} \boldsymbol{E}(\boldsymbol{k}_{\tau'}, \omega)) \right], \quad (2)$$

$\boldsymbol{\tau}' = 0, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots$.

The set of equations (2) describes the situation when a distributed feedback is formed by many diffracted waves (it is the analogy of the case of multiwave x-ray diffraction in crystals [13]). However, the analysis of such a general situation is very complicated. So we only consider here two-wave distributed feedback. This allows us to obtain all the main characteristics of x-ray FELS analytically and also to show the advantages of three-dimensional geometry of distributed feedback in comparison with the one-dimensional case.

So let us consider specifically the generation of a σ-polarized wave, i.e. the wave polarized to the diffraction plane. For the geometry of the so-called two-beam diffraction [14], where two strong waves are excited and diffraction occurs by the set of crystallographic planes, determined by a reciprocal lattice vector. In this case we can obtain a set of Maxwell's equations describing two-wave diffraction in a the crystal, having a beam penetrating through it, in the following way

$$\left(k^2 c^2 - \omega^2 \varepsilon_0 + \frac{\omega_L^2}{\gamma} + \frac{\omega_L^2}{\gamma} \frac{(\boldsymbol{u} \boldsymbol{e}'_\sigma)^2}{c^2} \times \frac{k^2 c^2 - \omega^2}{(\omega - \boldsymbol{k}\boldsymbol{u})^2} \right) E_\sigma - \omega^2 \varepsilon_\tau E_\sigma^\tau = 0$$
$$-\omega^2 \varepsilon_{-\tau} E_\sigma + \left(\kappa_\tau^2 c^2 - \omega^2 \varepsilon_0 + \frac{\omega_L^2}{\gamma} + \frac{\omega_L^2}{\gamma} \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{e^2} \times \frac{k_\tau^2 c^2 - \omega^2}{(\omega - \boldsymbol{k}_\tau \boldsymbol{u})^2} \right) E_\sigma^\tau = 0. \quad (3)$$

In (3)

$$E_\sigma = \boldsymbol{E}(k, \omega) \cdot \boldsymbol{e}_\sigma$$
$$E_\sigma^\tau = \boldsymbol{E}(\boldsymbol{k} + \boldsymbol{\tau}, \omega) \cdot \boldsymbol{e}_\sigma \qquad \boldsymbol{e}_\sigma \| [\boldsymbol{k}\boldsymbol{\tau}]$$

$\omega_L^2 = 4\pi e^2 n_0 / m$, where n_0 is the average electron (positron) density in a beam. Comparing (3) with the standard equation of x-ray dynamical diffraction we can see that the system of a crystal and particle beam may be considered as an active medium with dielectric susceptibility:

$$\tilde{\varepsilon}_0(\boldsymbol{k}, \omega) - 1 = \varepsilon_0 - 1 - \frac{\omega_L^2}{\gamma \omega^2} - \frac{\omega_L^2}{\gamma \omega^2} \times \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{c^2} \frac{k^2 c^2 - \omega^2}{(\omega - \boldsymbol{k}\boldsymbol{u})^2} \qquad \tilde{\varepsilon}_\tau = \varepsilon_\tau$$

$$\tilde{\varepsilon}_0(\boldsymbol{k}_\tau, \omega) - 1 = \varepsilon_0 - 1 - \frac{\omega_L^2}{\gamma \omega^2} - \frac{\omega_L^2}{\gamma \omega^2} \times \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{c^2} \frac{k_\tau^2 c^2 - \omega^2}{(\omega - \boldsymbol{k}_\tau \boldsymbol{u})^2} \qquad \tilde{\varepsilon}_{-\tau} = \varepsilon_{-\tau}.$$

Further, we shall analyse the generation of the wave with a wavevector $\boldsymbol{k}$, which makes a small angle with the particle velocity vector $\boldsymbol{u}$. In this case the wavevector $\boldsymbol{k}_\tau = \boldsymbol{k} + \boldsymbol{\tau}$ is directed at a large angle relative to $\boldsymbol{u}$, and consequently the magnitude of $(\omega - \boldsymbol{k}_\tau \boldsymbol{u})$ cannot become small. As a result, the terms containing an expression $(\omega - \boldsymbol{k}_\tau \boldsymbol{u})$ in their denominators will be small and can be ignored. We shall also neglect the term ω_L^2/γ—this is justified for real beam densities. It is well known that in order to provide non-zero solutions for the equation set (3), its determinant should be equal to zero. This also defines the dispersion equation, and for the σ-polarized wave it can be written in the form:

$$(\omega - \boldsymbol{k}\boldsymbol{u})^2 [(k^2 c^2 - \omega^2 \varepsilon_0)(k_\tau^2 c^2 - \omega^2 \varepsilon_0) - \omega^4 \varepsilon_\tau \varepsilon_{-\tau}] = -\frac{\omega_L^2}{\gamma} \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{c^2} \times (k^2 c^2 - \omega^2)(k_\tau^2 c^2 - \omega^2 \varepsilon_0). \quad (4)$$

The dispersion equation in such a form was derived in [11]. To solve the boundary problem, we use field continuity, the beam density and beam current density at the boundaries. For the latter two conditions we apply the following expressions, obtained from the equations of particle motion and the expression for the particle beam current:

$$j_\sigma = \frac{\mathrm{i} e^2 n_0}{m \gamma \omega} \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{c^2} \frac{k^2 c^2 - \omega^2}{(\omega - \boldsymbol{k}\boldsymbol{u})^2} E_\sigma$$
$$j_{\mathrm{d}v} - (\boldsymbol{u} \boldsymbol{e}_\sigma) n = -\frac{\mathrm{i} e^2 n_0}{m \gamma} \frac{(\boldsymbol{u} \boldsymbol{e}_\sigma)^2}{c^2 (\omega - \boldsymbol{k}\boldsymbol{u})} E_\sigma. \quad (5)$$

The dispersion equation (4) is sixth-order and, hence six solutions correspond to it. However, the two solutions corresponding to specularly reflected waves can be neglected due to the small value of $\varepsilon_\tau(\omega)$ in the x-ray region, $\varepsilon_0 - 1 \equiv g_0$; $\varepsilon_\tau \sim 10^{-5}$. Small values of ε_τ and g_0 will also be utilized when performing joining (we only join electric field strengths at boundaries).

A general solution for the field in a crystal is written as

$$E = \sum_{i=1}^{4} e_\sigma c_i \exp(\mathrm{i}\, k_i r)[1 + s_i \exp(\mathrm{i}\tau\tau)]$$

where k_i is the ith solution of the dispersion equation (4), and $k_{i\tau} = k_i + \tau$; τ is the reciprocal lattice vector corresponding to the planes of a diffraction reflection.

Taking into account the remarks made, the boundary conditions may be written as:

$$\begin{aligned} &c_1 + c_2 + c_3 + c_4 = 1 \\ &f_1 c_1 + f_2 c_2 + f_3 c_3 + f_4 c_4 = 0 \\ &g_1 c_1 + g_2 c_2 + g_3 c_3 + g_4 c_4 = 0 \\ &s_1 c_1\, \mathrm{e}^{\mathrm{i}K_{1z}L} + s_2 c_2\, \mathrm{e}^{\mathrm{i}K_{2z}L} \\ &\quad + s_3 c_3\, \mathrm{e}^{\mathrm{i}K_{3z}L} + s_4 c_4\, \mathrm{e}^{\mathrm{i}K_{4z}L} = 0 \end{aligned} \qquad (7)$$

$$s_i = \frac{\omega^2 \varepsilon - \tau}{k_{i\tau}^2 c^2 - \omega^2 \varepsilon_0} \qquad f_i = \frac{(u e_\sigma)^2}{(\omega - k_i u)}$$

$$g_i = \frac{k_i^2 c^2 - \omega^2}{(\omega - k_i u)^2} \frac{(u e_\sigma)^2}{c^2}.$$

In equations (7), only the boundary conditions are written which determine the field inside a crystal. The first equation corresponds to the continuity of an incident wave at the boundary $z = 0$; the second and third conditions correspond to the equality (5) at the entrance crystal to zero. The latter condition corresponds to zero equality of a diffracted wave at the exit boundary $z = L$ (we consider the Bragg diffraction geometry). k_i and k_{iz} ($i = 1/4$) are solutions of the dispersion equation (4). The linear system (7), defining the coefficients c_i placed ahead of i-modes in (6), has the solution $c_i = \Delta_i/\Delta$, where Δ is the determinant of the system (7), Δ_i is the ith minor, obtained as a result of replacement of the ith column by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that at $\Delta \to 0$ field amplitudes inside a crystal will increase, and thus the situation arises that when the field is non-zero when the incident wave is equal to zero. The $\Delta = 0$ condition with $\Delta_i \neq 0$ is called a generation threshold condition [15]. Substituting the expressions

$$\begin{aligned} &k_i = k_0 + k\delta_i n \qquad k_{0z} = \frac{\omega - k_\perp u_\perp}{u_z} \\ &k = \omega/c \qquad \delta_i \ll 1 \end{aligned} \qquad (8)$$

(where n is the normal to the crystal surface, $k_0 = (k_{0_z}, k_\perp)$) into the determinant Δ we can represent the generation threshold condition $\Delta = 0$ as

$$\begin{aligned} &\frac{(\delta_1 - \delta_2)(\delta_1 - \delta_3)(\delta_2 - \delta_3)}{\delta_1^2 \delta_2^2 \delta_3^2} s_4\, \mathrm{e}^{\mathrm{i}k\delta_4 L} \\ &- \frac{(\delta_1 - \delta_2)(\delta_1 - \delta_4)(\delta_2 - \delta_4)}{\delta_1^2 \delta_2^2 \delta_4^2} s_3\, \mathrm{e}^{\mathrm{i}k\delta_3 L} \\ &+ \frac{(\delta_1 - \delta_3)(\delta_1 - \delta_4)(\delta_3 - \delta_4)}{\delta_1^2 \delta_3^2 \delta_4^2} s_2\, \mathrm{e}^{\mathrm{i}k\delta_2 L} \\ &- \frac{(\delta_2 - \delta_3)(\delta_2 - \delta_4)(\delta_3 - \delta_4)}{\delta_2^2 \delta_3^2 \delta_4^2} s_1\, \mathrm{e}^{\mathrm{i}k\delta_1 L} = 0. \end{aligned} \qquad (9)$$

When writing (9) in expressions for f_i and g_i, non-resonance terms were neglected.

Further investigation will be based on a common consideration of equations (4) and (9). We substitute (8) into (4) and transform the dispersion equation as follows:

$$\begin{aligned} &\frac{(un)^2}{c^2} \delta^2 [4\gamma_0 \gamma_1 \delta^2 + 2(\gamma_1 l + \gamma_0 l_\tau)\delta + l l_\tau - \tau] \\ &= -\frac{1}{\gamma} \left(\frac{\omega_L}{\omega}\right)^2 \theta^2 \sin^2\varphi (l + g_0) l_\tau. \end{aligned} \qquad (10)$$

In (10) $\theta = k \wedge u$; $\varphi = k_{\perp_u} \wedge \tau_{\perp_u}$, where the $\perp_u$ sign denotes projection on a plane perpendicular to the velocity, $r = \varepsilon_\tau \varepsilon_{-\tau}$; $l = \theta^2 + g_0 + \gamma^{-2}$; $l_\tau = l + \alpha$; $\alpha = (2k_0\tau + \tau^2)/k^2$ is the departure from the Bragg condition.

$$\gamma_0 = \frac{(k_0 h)}{K} \qquad \gamma_1 = -\frac{(k_{0\tau} n)}{K}$$

are cosines of the angles, made by wavevectors of the transmitted and diffracted waves with the normal vector.

3. High-gain regime

It is known that in Bragg geometry, two solutions of the diffraction dispersion equation for a crystal without a beam have various signs for imaginary parts. The + sign in front of the imaginary part of one of these solutions corresponds to the absorption of a transmitted wave, and the − sign stands for absorption of a diffracted wave, taking into account the presence of an ultra-relativistic beam. Then, in the case of a proper choice of free parameters in the crystal–beam system, the imaginary part of the first solution may become negative, and this corresponds to the gain of a transmitted wave.

Thus, as follows from the expression for wave (6), to observe the gain in the Bragg case it is necessary to have two or more roots of the dispersion equation, which results in the growing exponents in Δ. The analysis shows that a high-gain regime is realized at a small deviation from a diffraction solution. That is why, in this region, it may be assumed

$$\mathrm{Re}(ll_\tau)=0$$
$$\Rightarrow l'=\mathrm{Re}\, l=\frac{-\alpha+\sqrt{\alpha^2+4r'}}{2}. \tag{11}$$

Equation (11) describes the diffraction branch in the absence of a beam and corresponds to the positive sign (+). Besides (11), there is one more diffraction branch, $l'=[-\alpha-(\alpha^2+4r')^{1/2}]/2$, but it is not in conformity with the Cerenkov synchronism condition for the x-ray region (due to a negative sign of $\mathrm{Re}\, g_0$). If (11) is satisfied, the dispersion equation (10) is rewritten as

$$\frac{(\boldsymbol{nu})^2}{c^2}\delta^2[4\gamma_0\gamma_1\delta^2+2(\gamma_1 l+\gamma_0 l_\tau)\delta - \mathrm{i}\varepsilon_0''(\sqrt{\alpha^2+4r'}-2\sqrt{r'})] = -\frac{1}{\gamma}\left(\frac{\omega_L}{\omega}\right)^2\theta^2\sin^2\varphi(l+g_0)l_\tau. \tag{12}$$

As a Cerenkov synchronism condition in the x-ray region does not coincide with the condition of anomalous transmission (the Borman effect) we do not consider lattice oscillations in the absorption term.

When the second term (in square brackets) in (12) may be written as

$$\delta_1=-(\gamma_0+\gamma_1)l'-\gamma_0\alpha+\mathrm{i}(\gamma_0+\gamma_1)\varepsilon_0''-\mathrm{i}\gamma_0\gamma_1\varepsilon_0''\times\frac{\sqrt{\alpha^2+4r'}-2\sqrt{r'}}{(\gamma_0+\gamma_1)\sqrt{\alpha^2+4r'}+(\gamma_0-\gamma_1)\alpha}$$
$$\delta_2=\frac{a}{2}(1-\mathrm{i}\sqrt{3})\qquad \delta_3=\frac{a}{2}(1+\mathrm{i}\sqrt{3})$$
$$\delta_4=-a \tag{13}$$

where

$$a=\left[\frac{1}{\gamma}\left(\frac{\omega_L}{\omega}\right)^2\theta^2\sin^2\varphi\,\frac{(l'+g_0')l_\tau'}{2(\gamma_1 l'+\gamma_0 l_\tau')}\frac{c^2}{(\boldsymbol{un})^2}\right]^{1/3}. \tag{14}$$

The first root δ_1 has a purely diffractive nature and corresponds to the mode which is not synchronized with the beam. The other roots, δ_2, δ_3, δ_4, are those depending on beam parameters and corresponding to the waves synchronized with a beam. One of these roots corresponds to a growing exponent (δ_2). Analysis shows that, to realize a threshold condition, δ_1 needs to be a diffracted wave and δ_2 a transmitted one. It should be noted that, when we assert that a solution falls into a transmitted wave or into a diffracted one, we bear in mind asymptotic behaviour. It is easy to see that, to bring the root δ_1 to $\mathrm{Im}\,\delta_1<0$, the following condition should be satisfied:

$$\gamma_1 l'+\gamma_0 l_\tau'>0. \tag{15}$$

In a high-gain regime we only take into account growing exponents, corresponding to roots δ_1 and δ_2. Then, from (9), it follows that

$$\Delta\sim(\delta_1-\delta_3)(\delta_1-\delta_4)\delta_2^2 s_2\,\mathrm{e}^{\mathrm{i}K\delta_2 L}-(\delta_2-\delta_3)(\delta_2-\delta_4)\delta_1 s_1\,\mathrm{e}^{\mathrm{i}K\delta_1 L}.$$

Substituting solutions δ_1, δ_2, δ_3 and δ_4 from (13) by Δ, Δ_1 and Δ_2 we obtain

$$E=\frac{(s_2-s_1)\exp[\mathrm{i}K(\delta_1+\delta_2)]}{s_2\exp(\mathrm{i}K\delta_2 L)-3s_1\exp(\mathrm{i}K\delta_1 L)}. \tag{16}$$

When the beam parameters correspond to the amplification region $|\mathrm{Im}\,\delta_2<|\mathrm{Im}\,\delta_1|$, i.e. the transmitted wave being in resonance with the beam, the generation of radiation will be less intensive than absorption in the diffracted wave, which is not in resonance with the beam. In this case a convective instability is developed [16] and a stationary process is possible.

For a sufficiently large length L of a crystal, in the region of amplification the amplitude of a transmitted wave has the asymptotic form (as follows from equation (16)):

$$|E|=\left|\frac{s_2-s_1}{3s_1}\right|\exp(-K\,\mathrm{Im}(\delta_2)\mathrm{L}). \tag{17}$$

We should note that, in the case of a cold beam, amplification can be observed despite the fact that attenuation in a crystal is large (the third summand in square brackets in (12) is much larger than the others). The increment changes its functional dependence on the beam density:

$$\delta_2=-\frac{b}{\sqrt{2}}(\mathrm{I}+i)$$
$$b=\left[\frac{1}{\gamma}\frac{c^2}{(\boldsymbol{un})^2}\left(\frac{\omega_L}{\omega}\right)^2\frac{\theta^2\sin^2\varphi(l'+g_0')l_\tau'}{\varepsilon_0''(\sqrt{\alpha^2+4r'}-2\sqrt{r'})}\right]^{1/2}. \tag{18}$$

As is seen from (18), at large absorption, the increment functional dependence on the beam density takes on the form $n_0^{1/2}$, rather than $n_0^{1/3}$ in (14). However, it is easy to see that under the circumstances, the increment corresponding to radiation production is always smaller than absorption in a diffracted wave. That is why, in this case (large absorption), the generation threshold cannot be reached (in the given formulation of the problem).

Let us now pass to the analysis of a transition from the convective instability to the absolute one in a high-gain case. For this purpose, we first specify a transition point, which is called a generation threshold [15]. It is determined by the requirement of equality of the determinant Δ to zero with the non-zero value of a corresponding numerator. As follows from (16), the generation threshold is determined by

$$s_2\, e^{iK\delta_2 L} - 3s_1\, e^{iK\delta_1 L} = 0. \qquad (19)$$

Prior to solving (19) let us determine the region of permissible parameters:

$$\gamma^{-2} + |g_0'| + \theta^2 = \frac{-\alpha + \sqrt{\alpha^2 + 4r'}}{2}. \qquad (20)$$

From (20) there follows a limitation of the Bragg parameter:

$$\alpha < \frac{r' - |g_0^p|^2}{|g_0^p|}. \qquad (21)$$

In (21) the following notation $|g_0^p| = |g_0'| + \gamma^{-2}$ is introduced. The quantity α has one more limitation, following from (15):

$$\alpha < \frac{\gamma_0 + \gamma_1}{\sqrt{-\gamma_0\gamma_1}}\sqrt{z'}. \qquad (22)$$

Condition (21) expresses the requirement of synchronism of root δ_2 with the Cerenkov condition and (22) is the condition that root δ_1 belongs to the diffracted wave (see above). Equation (19) should be rewritten as

$$-3\frac{\gamma_0}{\gamma_1}\frac{\sqrt{\alpha^2 + 4r'}}{\sqrt{\alpha^2 + 4r'} - \alpha}\, e^{iK\delta_1' L}\, e^{-K\delta_1'' L}$$
$$= \exp\left(i\frac{\omega ab}{2c}\right)\exp\left(\frac{ka}{2}\sqrt{3}b\right) \qquad (23)$$

where $\delta_1' \equiv \mathrm{Re}\,\delta_1$ and $\delta_1'' \equiv \mathrm{Im}\,\delta_1$. The phase condition from (23) provides $K\delta_1' L \simeq 2\pi n$, where n is an integer. Expressing δ_1'' in terms of α and using (11) we obtain from the phase condition:

$$\alpha = \frac{2\pi n}{kL}(\gamma_0 - \gamma_1) - (\gamma_0 + \gamma_1)\sqrt{\frac{4\pi^2 n^2}{k^2L^2} - \frac{r'}{\gamma_0\gamma_1}}. \qquad (24)$$

Equality (23), with (24), leads to the amplitude condition which determines the generation threshold:

$$\frac{\sqrt{3}}{2}kL\left(\frac{(\omega_L/\omega)^2 \sin^2\varphi}{2\gamma}(-\beta)r'\right.$$
$$\times \{[y + (x/2\beta) - (|g_0^p|/\sqrt{r'})][y + (x/2\beta)$$
$$\left.- (|g_0'|/\sqrt{z'})][y - (x/2\beta)]\}/x\right)^{1/3}$$
$$= K\varepsilon_0''\frac{L}{-\beta}\frac{(1-\beta)[(1+\beta)^2 y + (1+\beta)(x/2)] - 4\beta\sqrt{r'}}{x}$$
$$+ \gamma_0 \ln\left(3\frac{y - (x/2\beta)}{y + (x/2\beta)}\right). \qquad (25)$$

In (25)

$$y = \left[\left(\frac{x}{2\beta}\right)^2 - \frac{1}{\beta}\right]^{1/2}$$

$$x = -\frac{n}{(k\sqrt{r'}4)/(4\pi\gamma_0\beta)}$$

is the dimensionless number of a harmonic; $\beta = \gamma_1/\gamma_0$ is the asymmetry factor. Limitations (21), (22) are rewritten in the following way

$$0 < x < \frac{\sqrt{r'}}{|g_0^p|} + \beta\frac{|g_0^p|}{\sqrt{r'}} \qquad 0 < |\beta| < \frac{r'}{|g_0^p|^2}.$$

Expression (25) has rather a clear physical meaning. On the left-hand side there is the term associated with radiation gain in the case of the resonance interaction of a transmitted wave (Cerenkov synchronism is satisfied for transmitted wave) with a particle beam; the first summand on the right-hand side corresponds to radiation absorption in a diffracted non-resonant wave; the second term arises due to the escape of radiation through the surfaces of a crystal. In the high-gain regime the second summand on the right-hand side is much smaller than the first one. As follows from (25), with a decrease in the asymmetry factor $|\beta|$ absorption increases sharply. As the increment at $\beta = -r'/|g_0^p|^2$ is equal to zero, so the optimum conditions for reaching generation threshold will be satisfied in the region $|\beta| < r'/|g_0^p|^2$. The second summand in (25) decreases at an oblique incidence angle of the particle beam. This occurs because the particle beam interacts with the crystal across the length $\simeq 4/\gamma_0$. From (25), it follows that the generation threshold in a high-gain regime may be reached only at a large harmonic value, $n \sim (k\sqrt{r'}L)/(\gamma_0|\beta|)$. The harmonic number characterizes the distance between the diffraction roots, so, in the case of high gain, the generation threshold can only be realized far from the point of the root's degeneration.

We should note, in addition, that the generation condition obtained coincides with the condition for absolute instability, taking account of reflection from the target boundaries. It is known that this condition reduces to the equality $\mathrm{Im}\,k_+(\omega) = \mathrm{Im}\,k_-(\omega)$ [17] where L_+ corresponds to the wave propagating forward and k_- to the reflected wave. If the solution of this equation has $\mathrm{Im}\,\omega < 0$, then the subthreshold situation is observed. At $\mathrm{Im}\,\omega > 0$ absolute instability is realized; $\mathrm{Im}\,\omega = 0$ corresponds to the generation threshold. In our case the role of a reflected wave is played by a diffracted wave, and the condition (19) in the limit of a large crystal length takes the form $\mathrm{Im}\,\delta_1 = \mathrm{Im}\,\delta_2$. The equality of radiation generation and its attenuation corresponds to the threshold. The numerical estimation, based on (25), gives the threshold value of $j \sim 10^7$–$10^8\,A$ cm^{-2} (depending on the angle of beam incidence on a crystal) at $j = 1500$.

4. Low-grain regime

Let us now turn to the consideration of instability behaviour in the opposite limit ($k|\mathrm{Im}\,\delta|L < 1$). In this case the generation threshold is observable at some deviation from the Cerenkov condition. Solutions of the dispersion equation (4) can be classified into diffraction and beam

$$\delta_{1,2} = a \pm \sqrt{a^2+b+(c_0/x_{1,2}^2)}$$

$$\delta_{3,4} = \pm \mathrm{i}\sqrt{\frac{c_0}{b}} + a\frac{c_0'}{b^2}$$

$$a = -\frac{1}{4}\left(\frac{l}{\gamma_0}+\frac{l_\tau}{\gamma_1}\right) \qquad b = -\frac{ll_\tau - r}{4\gamma_0\gamma_1} \tag{26}$$

$$x_{1,2} = a \pm \sqrt{a^2+b}$$

$$c_0 = -\frac{(\omega_L/\omega)^2}{4\gamma_0\gamma_1\gamma}\frac{c^2}{(\boldsymbol{un})^2}\theta^2\sin^2\varphi(1+g_0)l_\tau.$$

Let us substitute (26) into the expression for the generation threshold condition (9). Then after some calculation we obtain the expression for the generation threshold in the form:

$$-\frac{\pi^2 n^2}{4\gamma}\left(\frac{\omega_L}{\omega}\right)^2 k^3 L_*^3\left(\frac{|\varepsilon_\tau|}{\sqrt{-\beta}}+g_0'-\gamma^{-2}\right)$$
$$\times\left(\frac{|\varepsilon_\tau|}{\sqrt{-\beta}}+g_0'\right)\sin^2\varphi\sin y$$
$$\times\frac{(2y+\pi n)\sin y - y(y+\pi n)\cos y}{y^3(y+\pi n)^3}$$
$$=\left(\frac{\gamma_0 c}{(\boldsymbol{un})}\right)^3\frac{16(-\beta)\pi^2 n^2}{(k|\varepsilon_\tau|L_*)^2} + k\varepsilon_0'' L_*\left(\frac{1}{\sqrt{-\beta}}-1\right)^2 \tag{27}$$

where $y=(kx_2'L)/2$ and n is an integer; $L_* = L\,u/(\boldsymbol{un})$ is the distance a beam travels in the crystal. Following from (27), the generation threshold is reached in the vicinity of a degeneracy root point $(2\pi n)/(k\sqrt{r'}L) \ll 1$. It should be noted that, due to the requirement of Cerenkov synchronism in the x-ray region, the threshold will be reached at a small asymmetry factor value $|\beta| < r'/|g_0^g|^2$. Owing to this, one-dimensional geometry (backward scattering) in this case, in principle, is not realized.

5. Hot-beam limit

We have hitherto considered an ideal monochromatic particle beam (cold limit). Now we shall turn to the opposite limiting case of a 'hot beam'. The criterion for a beam may be easily obtained from the estimation of the destruction of Cerenkov synchronism due to the velocity spread $\Delta(\omega - \boldsymbol{ku}) \simeq k\delta u_z + \boldsymbol{k}_\perp\Delta\boldsymbol{u}_\perp + K_\parallel\Delta u_z$. Hence it appears that, when satisfying the condition $|\delta| < \theta(\psi_1\cos\varphi + \psi_2\sin\varphi + \psi_\parallel)$, the relativistic particle beam should be considered as a hot beam. Here ψ_1, ψ_2, $\psi_\parallel$ are the two transverse and longitudinal angular beam spreads respectively; θ is the angle of radiation $\varphi = \boldsymbol{k}_\perp{}^\wedge \boldsymbol{l}$, where $\boldsymbol{l}$ characterizes the direction in which the spread is equal to ψ_1. In a 'hot beam' case it is necessary to introduce a distribution function. The equations of particle motion in this case are replaced by the equation for the distribution function:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \boldsymbol{r}} + e\left[\boldsymbol{E} + \frac{1}{c}(\boldsymbol{vH})\right]\frac{\partial f}{\partial \rho} = 0 \tag{28}$$

and the current density is $\boldsymbol{j} = e\int \boldsymbol{v}f\,\mathrm{d}\boldsymbol{p}$.

After integration of the current density over the distribution function, the resonance Cerenkov term in this limit becomes smooth and the dispersion equation takes the form

$$(k^2c^2 - \omega^2\varepsilon_0 + \mathrm{i}\omega^2\Gamma)(k_\tau^2c^2 - \omega^2\varepsilon_0) - \omega^4 r = 0. \tag{29}$$

We assume for simplicity $\psi_1 = \psi_2 = \psi_\perp$ (this does not restrict the problem being considered, as diffraction geometry should be chosen in such a way that the influence of a particle beam spread on the synchronism condition would be minimum). The longitudinal velocity spread will not be taken into account (it starts to play a role only at small angles of radiation, but we do not analyse these angles as small angles correspond to small increments). Then one can use the distribution function in the form $f_0(\boldsymbol{v}) = (2\pi c^2\psi_\perp^2)^{-1}$.

$$\exp\left(-\frac{v_\perp^2}{2c^2\psi_\perp^2}\right)\delta(v_\parallel - u)$$

is the gain increment and is written as:

$$\Gamma = -\frac{\sqrt{\pi}}{\gamma}\left(\frac{\omega_L}{\omega}\right)^2\frac{\theta^2+\gamma^{-2}}{\psi_\perp^2}x\,\mathrm{e}^{-x^2} \tag{30}$$

where

$$x = \frac{\omega - \boldsymbol{ku}}{\sqrt{2}\omega\theta\psi_\perp}.$$

Equation (29) has four solutions but two of them correspond to specularly reflected waves and we do not consider them. As a result, the generation condition in this case is determined by the relation

$$s_2\,\mathrm{e}^{\mathrm{i}K\delta_2 L} - s_1\,\mathrm{e}^{\mathrm{i}K\delta_1 L} = 0. \tag{31}$$

Substituting (8) into (29) we get:

$$\delta_{1,2} = \frac{-\gamma_1(l+\mathrm{i}\Gamma_{1,2}) - \gamma_0 l_\tau \pm \{[\gamma_1(l+\mathrm{i}\Gamma_{1,2}) - \gamma_0 l_\tau]^2 + 4\gamma_0\gamma_1 r\}^{1/2}}{4\gamma_0\gamma_1} \tag{32}$$

If we now substitute (32) into (31), we obtain the following value for generation threshold:

$$\Gamma^* = \Gamma_{\mathrm{thr}}^* = \left(\frac{\gamma_0 c}{(\boldsymbol{un})}\right)^3\frac{16(-\beta)\pi^2 n^2}{(K\sqrt{r'}L_*)^2 kL_*} + \left(1 - \frac{1}{\sqrt{-\beta}}\right)^2\varepsilon_0'' \tag{33}$$

where

$$L = L_*\frac{(\boldsymbol{un})}{u}$$

$$\Gamma^* = -\frac{\sqrt{\pi}}{2\gamma}\left(\frac{\omega_L}{\omega}\right)^2\left(\sqrt{\frac{r'}{-\beta}} - |g_0'|\right)\Big/ \psi_\perp^2/g_0'(x_1\,e^{-x_1^2} + x_2\,e^{-x_2^2})\sin^2\varphi.$$

In the case of a high-gain regime and 'hot' beam, the amplitude condition for the threshold is written as:

$$\Gamma^* = 2\frac{(1-\beta)[(x/2\beta)^2-(1/\beta)]^{1/2}+2}{[(x/2\beta)^2-(1/\beta)]^{1/2}-(x/2\beta)}\frac{\varepsilon_0''}{(-\beta)}. \tag{34}$$

Functional dependence of gain on crystal length, in this case, is the same for a low- (33) and high-gain (34) regime. This is attributed to the fact that $\Gamma \sim n_0$.

A beam may be 'hot', not owing to the initial level of a velocity particle spread, but as a result of multiple scattering during crystal motion. In this case, the beam becomes even 'hotter' as it moves into the crystal depth, and the increment already depends on the length along the beam L_*. Such a problem may be studied in the Eikonal approximation [18]. Here, instead of the threshold condition (33), we get:

$$\int_0^L \Gamma^*\left(\frac{zu}{(\boldsymbol{un})}\right)\mathrm{d}z = \frac{16\gamma_0^3(-\beta)\pi^2 n^2}{[(w/c)\sqrt{r'}L]^2(\omega/c)} + \left[1-\frac{1}{\sqrt{-\beta}}\right]^2\varepsilon_0''L. \tag{35}$$

Taking into account the dependence of the particle velocity spread on a multiple scattering angle

$$\psi_\perp^2 = \psi_0^2 + \frac{E_s^2}{E^2}\frac{zc}{(\boldsymbol{un})L_{\mathrm{rad}}}$$

and that of the 'hot' beam increment on the velocity spread $\Gamma = \Gamma_0/\psi_\perp^2$, we get the equality for determination of a generation condition:

$$A\sqrt{\pi}\left(\frac{\omega_L}{\omega}\right)^2\left(\sqrt{\frac{r'}{-\beta}} - |g_0'|\right)\gamma\frac{m_e^2c^4}{E_s^2} \times \ln\left(1+\frac{E_s^2}{m_e^2c^4\gamma^2}\frac{L_*}{\psi_0^2L_{\mathrm{rad}}}\right) = \frac{16\gamma_0^3c^3}{(\boldsymbol{un})^3}\frac{(-\beta)\pi^2n^2}{(k\sqrt{r'}L_*)kL_{\mathrm{rad}}} + \left(1-\frac{1}{\sqrt{-\beta}}\right)^2\varepsilon_0''\frac{L_*}{L_{\mathrm{rad}}} \tag{36}$$

$A = \max(-x\,e^{-x^2}) \approx 0.4.$

At a large particle energy E, when the influence of multiple scattering may be ignored, i.e.

$$E_s^2L_*(m_e^2c^4\psi_\perp^2L_{\mathrm{rad}})^{-1} \ll 1$$

equation (36) transforms into (33). But the optimum conditions for reaching the generation threshold are satisfied when the length of the crystal plate is equal to the quantity at which the multiple scattering exerts a considerable influence. Numerical calculation shows that the length may be considered as optimal when

$$E_s^2L_*(m^2c^4\gamma^2\psi_0^2L_{\mathrm{rad}})^{-1} \simeq 4.$$

As follows from (36), with further increase in the crystal length, the condition for generation threshold become worse and this leads to the fact that the requirements for beam parameters become more rigid. From (36) it follows that, in the case of an optimal value of the asymmetry factor $\beta^o = -0.04$, the following requirement to the beam arises

$$\frac{A\sqrt{\pi}}{\gamma}\left(\frac{\omega_L}{\omega}\right)^2\frac{|\varepsilon_\tau|}{\psi_0^2\varepsilon_0''} \simeq 12,$$

which implies, for example for the beam with $\gamma = 1500$, $\psi_0 \sim 5\times10^{-6}$ rad, when radiation is diffracted by the surface (111) LiH, $j \sim 10^{10}$ A cm^{-2}. In the case of the influence of multiple scattering we obtain, proceeding from (36),

$\gamma^{\mathrm{opt}} \sim 20$ GeV $\quad L_{*\mathrm{opt}} \sim 10^{-2}$ cm $\quad j \sim 10^{12}$ A cm^{-2}.

6. Generation regime

We have discussed the subthreshold and threshold regimes of radiation. If parameters of the beam amplification are such that the generation threshold is exceeded, then the radiation growth at every point of the beam is observed, i.e. an absolute instability takes place. Let us determine the increment of an absolute instability corresponding to the case of a 'hot' beam. Rewriting (31), we shall bear in mind that in this case an imaginary term appears in the frequency itself. As a result, in the weak-amplification regime we have

$$k\frac{[(\gamma_1l'-\gamma_0l_\tau')^2+4\gamma_0\gamma_1r']^{1/2}}{2\gamma_0|\gamma_1|}L = 2\pi n$$

$$k\frac{(\gamma_1l'-\gamma_0l_\tau')[\gamma_1\Gamma^*-2\gamma_0(\gamma_1-\gamma_0)(c/(\boldsymbol{un})(\omega''/\omega')(\gamma_0-\gamma_1)\varepsilon_0'']+2\gamma_0\gamma_1r'}{2\gamma_0\gamma_1[(\gamma_1l'-\gamma_0l_\tau')^2+4\gamma_0\gamma_1r']^{1/2}}L + \frac{[(\gamma_1l'-\gamma_0l_\tau')^2+4\gamma_0\gamma_1r']^{1/2}}{\gamma_0l_\tau} = 0. \tag{37}$$

With the aid of (33) this expression may be rewritten as

$$\omega'' = \omega'\frac{-\beta}{2(1-\beta)}(\Gamma^* - \Gamma_{\mathrm{thr}}) \tag{38}$$

where Γ_{thr}^* is defined by the right-hand side of (33).

In the low-gain regime, for the cold beam the increment of an absolute instability may be obtained in the same way. In this case the expression takes the form:

$$\omega'' = \omega'\frac{-\beta}{2(1-\beta)}\Big[-\pi^2n^2\frac{(\omega_L/\omega)^2}{4\gamma}\sin^2\varphi k^2L_*^2 \times\left(\frac{|\varepsilon_\tau|}{\sqrt{-\beta}}+g_0'-\gamma^{-2}\right)\left(\frac{|\varepsilon_\tau|}{\sqrt{-\beta}}+g_0'\right)\Big] \times \sin y\frac{(2y+\pi n)\sin y - y(y+\pi n)\cos y}{y^3(y+\pi n)^3} - \frac{16(-\beta)\pi^2n^2}{(k|\varepsilon_\tau|L_*)^2kL_*}\left[\frac{\gamma_0}{\boldsymbol{un}}\right]^2 - \varepsilon_0''\left(\frac{1}{\sqrt{-\beta}}-1\right)^2. \tag{39}$$

It follows from expressions (38) and (39) that, for the 'hot' $10^9\,s^{-1}$ beam with $\psi_\perp = 5\times10^{-6}$ and $j \sim 5\times10^{10}$ A cm^{-2}, an absolute increment will be $\omega'' \sim 10^9\,s^{-1}$. The same value for increment is obtained in the case of the cold beam ($\psi_\perp < 10^{-7}$) with current density $j \sim 5\times10^9$ A cm^{-2} at the length $L_* \sim 1$ mm.

7. Conclusion

We have considered the x-ray FEL based on parametric (quasi-Cerenkov) radiation. It has been shown that only dynamic x-ray diffraction gives rise to the conditions under which the mechanism of radiation in the x-ray region appears and a distributed feedback is realized, which considerably decreases the required beam density. As we can see, various regimes of x-ray parametric FEL operation are possible.

We should emphasise the great importance of the three-dimensional geometry of dynamic x-ray diffraction considered in this paper: one-dimensional diffraction does not allow one to reach, in principle, the threshold in the low-gain regime. Three-dimensional diffraction also provides a considerable decrease in the required beam densities in any regime operating under conditions of a small asymmetric factor of diffraction.

Although these densities remain rather large, they are considerably less than those in the one-dimensional case [2, 3]. Moreover, the analysis of the amplification gain in the multiwave diffraction case [10, 11] and the calculation of threshold beam parameters in the coplanar three-wave diffraction show that the transition to the multiwave distributed feedback in the crystal resonator leads to a further decrease in the threshold value of the particle beam density. Consequently it is necessary to form the distributed feedback in a parametric quasi-Cerenkov generator by multiwave diffraction in a crystal resonator.

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